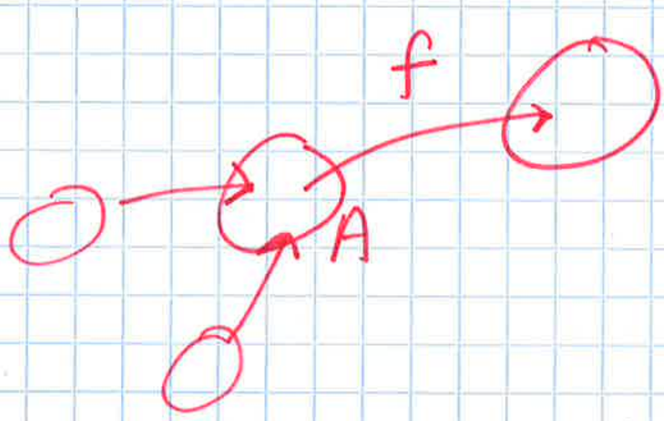
Some More Set Theory

Let $f: X \to X$ be a dynamical system. If $A \subseteq X$, then define

$$f(A) := \{ f(a) : a \in A \}$$

pre-image:

$$f^{-1}(A) := \{ x \in X : f(x) \in A \}$$



understand dynamics by looking at "invariants"

A set A is called <u>invariant</u> if

$$f(A) \subset A.$$

the set A is called <u>backward invariant</u> if

$$f^{-1}(A) \subset A.$$

Recall/Notation

the orbit of x

$$\mathcal{O}_+(x) = \{f^k(x) : k \geq 0\}$$
$$= \{x, f(x), f^2(x), \ldots\}$$

★ Theorem | A set A is invariant iff ("if and only if") for any $x \in A$ the orbit of x is a subset of A

$$\mathcal{O}_+(x) \subset A$$

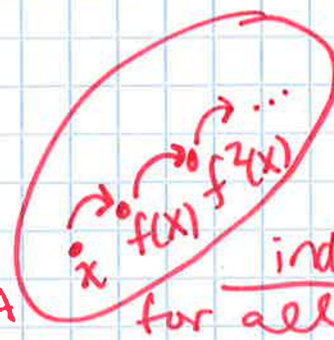
(ie A is <u>saturated</u> by orbits)

Proof of ☆

Aside: if and only if means that the ~~[illegible]~~ first statement implies the other AND the second statement implies the first.

We will first show that if A is invariant, then for every $x \in A$ $\mathcal{O}_+(x) \subset A$.

let $x \in A$.
base case



induction!

We prove that $f^k(x) \in A$ for all $k \geq 0$ by induction. The case when $k=0$ follows by assumption.

Assume that $f^k(x) \in A$. Then $f^{k+1}(x) = f(f^k(x))$. Since $f^k(x) \in A$ and A is invariant $f^{k+1}(x) \in A$.
(follows from definition).

other half is left as an excersise

Exercises

1] Show that the converse is true

ie Show that if A set A has the property that for every $x \in A$, $\mathcal{O}_+(x) \in A$, then A is invariant.

2] Show that every backward invariant set is (forward) invariant but not vice versa.

now for something totally different

Recall

In $\mathbb{R}/2\pi\mathbb{Z}$

$$[x] = \{\ldots, x-4\pi, x-2\pi, x, x+2\pi, \ldots\}$$

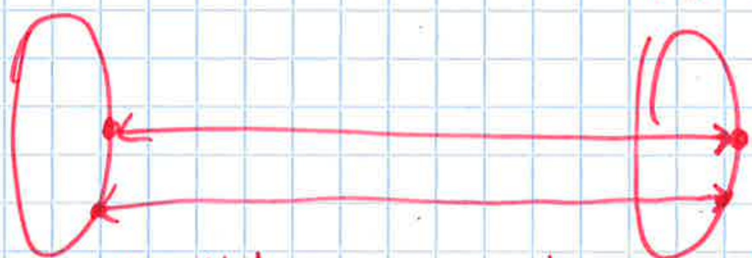
$$= \{x + k2\pi : k \in \mathbb{Z}\}$$

Define $[x]_{\mathbb{Z}} = \{x + k : k \in \mathbb{Z}\}$

So $\mathbb{R}/\mathbb{Z}$ is the set of these new $[\cdot]_{\mathbb{Z}}$ equivalence classes

The natural way to show that two spaces are the "same" is to find an (special) function between them.

$$H : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$



build a way to identify points to each other

want: a function that is both 1-1 and onto between these sets.

Define $H([x]_{\mathbb{Z}}) = [2\pi x]$

Step 1: H is well defined.

need to check that the definition only depends on the $[x]_{\mathbb{Z}}$ set and not the representative x of the set. So if I choose a diff. point in the set I should get the same equiv class

Suppose that

$[x]_{\mathbb{Z}} = [y]_{\mathbb{Z}}$. Need to check

that $[2\pi x] = [2\pi y]$.

Since $[x]_{\mathbb{Z}} = [y]_{\mathbb{Z}}$,

$y = x + k$ for some $k \in \mathbb{Z}$.

$\downarrow$ multiply by 2π

Hence,

$2\pi y = 2\pi x + 2\pi k$

So $2\pi y \in [2\pi x]$ and $[2\pi y] \ni 2\pi x$

ie $[2\pi y] = [2\pi x]$:)

$H_{fake}([x]_{\mathbb{Z}}) = [3x]$ $\longleftarrow$ not well defined

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one way to show 1-1 and onto is to define an inverse function

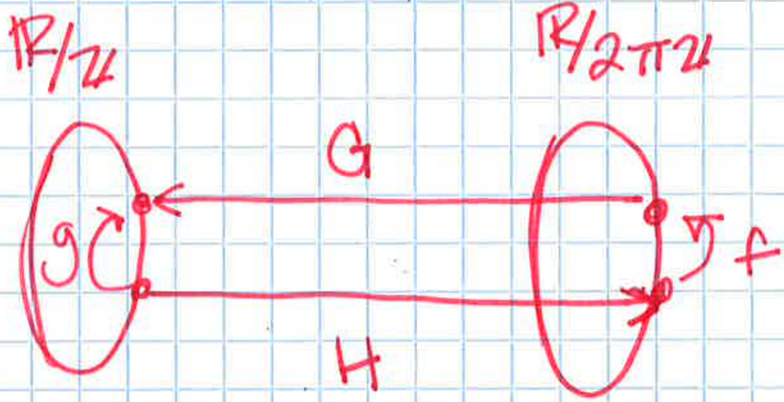
Define $G : \mathbb{R}/_{2\pi\mathbb{Z}} \to \mathbb{R}/_{\mathbb{Z}}$

$G([x]) = \left[\frac{x}{2\pi}\right]_{\mathbb{Z}}$

Exercise: show that G is well defined

Commutative diagram time!

How does this interact with dynamics?



$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[r, "g\,?"] \arrow[d, "H"'] & \mathbb{R}/\mathbb{Z} \arrow[d, shift left, "H"] \\ \mathbb{R}/2\pi\mathbb{Z} \arrow[r, "f"'] & \mathbb{R}/2\pi\mathbb{Z} \arrow[u, shift left, "G"] \end{tikzcd}$$

↑ dynamics

$g = \cancel{G} H^{-1} \circ (f \circ H)$

↓ nicer version

$H(g([x]_{\mathbb{Z}}))$

$= f(H([x]_{\mathbb{Z}}$

when H is invertible, f and g are <u>conjugated</u> by H and think of them as "the same" system.